Duality of Plane Curves

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Abstract

The *dual* of an algebraic curve C in $\mathbb{RP}^2$ defined by the polynomial equation $f(x,y,z) = 0$ is the locus of points $\left(\frac{\partial f}{\partial x}(a,b,c) : \frac{\partial f}{\partial y}(a,b,c) : \frac{\partial f}{\partial z}(a,b,c)\right)$ where $(a:b:c) \in C$. The dual can alternatively be defined geometrically as the image under reciprocation of the envelope of tangent lines to the curve. It is known that the dual of an algebraic curve is also an algebraic curve, and that taking the dual twice results in the original curve. We use reciprocation to obtain an elementary geometric proof of the latter fact, and we use our methods to gain intuition about the relationship between a curve and its dual.

1 Introduction

The notion of duality plays a central role in projective geometry. In particular, the notions of "point" and "line" can be interchanged in any self-dual projective plane, enabling one to dualize any theorem about points in the plane to obtain a "dual theorem" about lines.

The use of this duality dates back to Appolonius, but it was not until much later that it was generalized to plane curves of higher degree. In the 1830's, Plücker first defined the dual of an *algebraic curve* in the projective plane, that is, a curve defined by a homogeneous polynomial equation $f(x,y,z) = 0$, to be the envelope of all tangent lines to the curve. These tangent lines can each be described by a triple of homogeneous coordinates $(a : b : c)$ corresponding to the equation $ax+by+cz = 0$. It turns out that the resulting set of triples is also the set of solutions to some homogeneous polynomial equation $g(a,b,c) = 0$. Thus we have the notion of a *dual curve*, which we denote C^*. [4], [5]

Remarkably, taking the dual of the dual of a curve results in the original curve: we have $(C^*)^* = C$. However, the dual of a nonsingular curve C defined by a polynomial f of degree d is defined by a polynomial g of degree exactly $d(d-1)$. It would seem at first glance that taking the dual of C^* should therefore result in a curve of yet higher degree, and so $(C^*)^*$ would have higher degree than C. This became known as the *duality paradox*. [4]

In 1839, Plücker resolved the duality paradox by showing that the degree of C^* is lowered with each singularity of C. In particular, it is lowered by 2 for each double point (node) and by 3 for each cusp. In particular, this implies that the dual of a nonsingular curve of degree greater than 2 must have a singular point, and in some cases C and C^* may have the same degree. [1]

In fact, it is possible for a curve to be projectively equivalent to its dual. That is, the polynomials f and g describing C and C^* respectively may be related by an equation of the form $f = g \circ \phi$ where ϕ is an invertible projective transformation. We say such a curve C is *self-dual.* The problem of classifying all self-dual algebraic curves in the real projective plane is currently open. [3].

In this paper, we aim to shed light on the geometric relationship between an algebraic curve and its dual, and explain some of the phenomena described above using purely geometric methods. In section 2, we describe several equivalent ways of viewing the real projective plane and give explicit isomoprhisms from the plane to its dual. In section 3, we use these notions to provide a geometric proof of the relation $(C^*)^* = C$. Finally, in section 4, we use our geometric understanding of duality to illustrate several examples of dual pairs of curves, including a self-dual cubic curve.

2 Background

As a starting point for understanding duality in projective geometry, we first recall the axioms defining a general projective plane.

Definition 1. A *projective plane* is a pair (P, L) where P is a nonempty set of *points* and L is a nonempty collection of subsets of P called *lines*, satisfying the following three axioms:

P1: Every pair of distinct points are contained in exactly one line.
P2: Every pair of distinct lines intersect in exactly one point.
P3: There are four distinct points $a, b, c, d \in P$ no three of which lie on a line.
P4: Every line contains at least three points.

One immediately notices that axioms **P1** and **P2** are similar in nature. In particular, if we rename "points" as "lines" and vice versa, and identify a point with the set of lines containing it, then axiom **P1** becomes **P2** and vice versa. This suggests an underlying duality between points and lines.

Definition 2. The *dual* of a projective plane (P, L), denoted P^*, is the pair (L, P), where P is viewed as a collection of subsets of L by associating each point with the set of lines containing it.

It is not hard to see that if (P, L) satisfies axioms **P1-P4**, then so does (L, P). Thus the dual of a projective plane is a projective plane.

The smallest example of a projective plane is known as the *Fano plane*, consisting of seven points and seven lines as in Figure 1.

Notice that we can bijectively map the points of the Fano plane F_7 onto the lines, by mapping point A to line a, B to b, and so on as labeled in the figure. This particular bijection, say ϕ, is an *isomorphism* of projective planes $F_7 \cong F_7^*$, as it *preserves incidence*: point M lies on line l if and only if the line $\phi(M)$ contains the point $\phi^{-1}(l)$.

Remark 1. Not all projective planes are self-dual. The smallest counterexample is the *Hall plane* of order 9, having 91 points and 91 lines, each line containing 9 points and each point lying on 9 lines as described in [6].

We now turn our attention to the real projective plane, which also exhibits a beautiful self-duality.

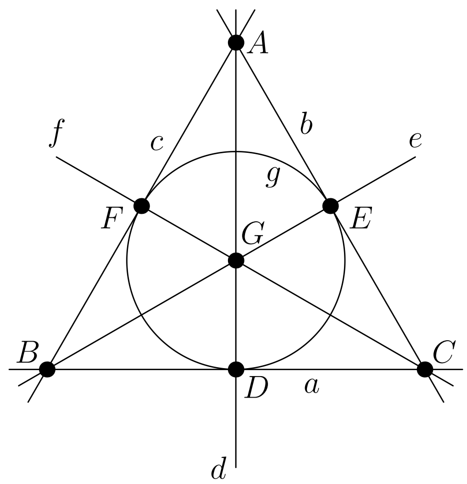

Figure 1: The Fano projective plane F_7.

2.1 The real projective plane

The real projective plane was first defined by Desargues as an extension of the Euclidean plane that forces parallel lines to intersect "at infinity", hence creating a plane that satisfies axiom **P2** rather than the parallel postulate. [4]

It may seem contradictory to require parallel lines to meet, but in fact it is a natural model of things we see in reality. Suppose you are standing on train tracks and looking out towards the horizon along the direction of the tracks. The tracks are parallel, but they appear to converge to a point on the horizon. In this sense, Euclidean geometry is simply a local version of projective geometry; parallel lines are only truly "parallel" near your feet!

Thus, to extend the Euclidean plane into a projective plane, one simply needs to add an extra point at infinity for each equivalence class of parallel lines in the plane. Define a *pencil of parallel lines* in the Euclidean plane $\mathbb{E}^2$ to be the set of all lines parallel to a given line in the plane.

Definition 3. The *real projective plane*, denoted $\mathbb{RP}^2$, is the pair $(\mathbb{E}^2 \sqcup l^\infty, L)$, where l^∞ the set of all pencils of parallel lines in the Euclidean plane $\mathbb{E}^2$, and where the collection of projective lines L consists of:

- l^∞, and
- all subsets of $\mathbb{E}^2 \sqcup l^\infty$ consisting of a line in $\mathbb{E}^2$ along with its pencil of parallel lines in l^∞, called its *point at infinity.*

It is easy to verify that this makes $\mathbb{RP}^2$ into a projective plane according to axioms P1-P4.

Just as the Euclidean plane can be defined either using the axioms of Euclidean geometry or Cartesian coordinates, the real projective plane can alternatively be defined using coordinates, known as *homogeneous coordinates.*

Figure 2: Parallel train tracks meeting at a point on the horizon.

Definition 4. The real projective plane $\mathbb{RP}^2$ is the set of all equivalence classes of nonzero points in $\mathbb{R}^3$ under scaling. That is,

$$\mathbb{RP}^2 = \left\{(x, y, z) \in \mathbb{R}^3 \mid (x, y, z) \neq (0, 0, 0)\right\} / \sim$$

where $\sim$ is the equivalence relation defined by $(x, y, z) \sim (a, b, c)$ if and only if there exists $\lambda \in \mathbb{R} \setminus \{0\}$ with $x\lambda = a$, $y\lambda = b$, and $z\lambda = c$. We write $(x : y : z)$ to denote the equivalence class associated to the point (x, y, z).

With the points defined in terms of coordinates, the *lines* in the real projective plane are defined as the loci of points $(x : y : z)$ in $\mathbb{RP}^2$ satisfying

$$ax + by + cz = 0$$

for some fixed $a, b, c \in \mathbb{R}$, not all zero. Again, it is easy to verify that this makes $\mathbb{RP}^2$ into a projective plane according to axioms P1-P4.

To show that these two definitions are equivalent, we can use a type of map projection known as *gnomonic projection*. We first identify $\mathbb{RP}^2$ with half of the unit sphere in $\mathbb{R}^3$. For a point given by homogeneous coordinates $(x : y : z)$, there exist exactly two values of λ for which $(x\lambda, y\lambda, z\lambda)$ is a point on the unit sphere in $\mathbb{R}^3$, and these two points are diametrically opposite each other on the unit sphere. Thus, we can interpret $\mathbb{RP}^2$ as the unit sphere having opposite points identified.

We now wish to choose one representative from each pair of diametrically opposite points on the unit sphere to form a half-sphere. In particular, we define S^+ to be the set of points (x, y, z) on the unit sphere having either:

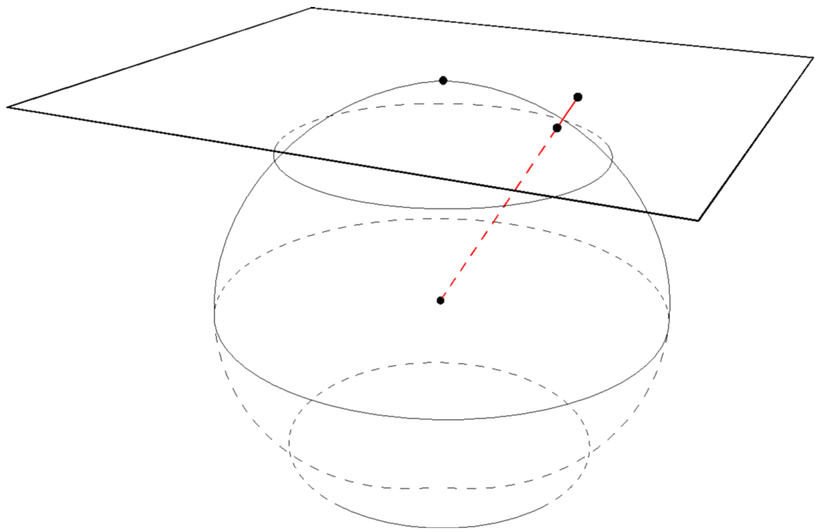

Figure 3: Gnomonic projection.

1. $z > 0$,
2. $z = 0$ and $y > 0$, or
3. $z = 0$, $y = 0$, and $x > 0$.

Clearly every pair of diametrically opposite points has exactly one of the points lying in S^+. Thus we can identify S^+ with (the coordinate definition of) $\mathbb{RP}^2$. Here, the lines are the arcs on S^+ cut out by great circles on the unit sphere, as the equations $ax + by + cz = 0$ define planes through the origin in $\mathbb{R}^3$.

Now, let S_z denote the upper half of this sphere consisting of the points having *positive* z-coordinate (case 1 above). We can project this half-sphere bijectively onto the Euclidean plane via *gnomonic projection*, as follows.

Definition 5. Let P denote the plane $z = 1$ in $\mathbb{R}^3$. The *gnomonic projection* (in the z direction) is the map $\phi : S_z \to P$ sending

$$(x, y, z) \mapsto (x/z, y/z, 1).$$

Geometrically, this corresponds to taking the point of intersection of the line through the origin passing through (x, y, z) on S with the plane P. The image of ϕ is the entire plane $z = 1$, and is hence isomorphic to the Euclidean plane $\mathbb{R}^2$. Considered as a subset of the entire projective plane $\mathbb{R}^2 \sqcup l^\infty$, the image of S_z is called the (zth) *affine patch* of the projective plane.

Finally, we can extend the map ϕ to a map $S^+ \to \mathbb{R}^2 \sqcup l^\infty$ by sending the half-circle at $z = 0$ to the line at infinity l^∞ in the natural way (so that incidence is preserved).

This gives the desired equivalence of the two notions of the real projective plane.

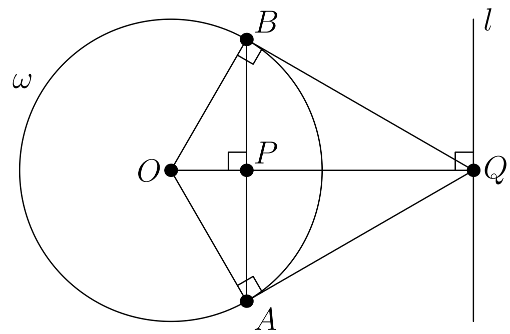

Figure 4: Reciprocation about a circle ω.

2.2 Duality in $\mathbb{RP}^2$

We now describe a class of isomorphisms from $\mathbb{RP}^2$ to its dual $(\mathbb{RP}^2)^*$ known as *reciprocation* in a conic. We begin by defining reciprocation in a circle.

Recall that an isomorphism from $\mathbb{RP}^2$ to its dual consists of a bijection from the set of points to the set of lines $\mathbb{RP}^2$ that preserves incidence: we require that point P lies on line l if and only if the image of P in the dual space is a line containing the image of l.

Definition 6. Let ω be any circle in $\mathbb{R}^2 \subset \mathbb{RP}^2 = \mathbb{R}^2 \sqcup l^\infty$, with center O and radius r. Let P be any point in the plane with distance $d > 0$ from O. The *reciprocal* of P about ω is defined to be the line l perpendicular to line OP, intersecting OP on the same side of O as P, having distance d' from O where

$$d' = r^2/d.$$

We also say that the reciprocal of l is P.

If $P = O$, the reciprocal of P is defined to be l^∞, and if P is a point on l^∞, its reciprocal is the line through O perpendicular to the pencil of parallel lines associated with P.

The reciprocal of a point is called its *polar*, and the reciprocal of a line is called its *pole*.

Remark 2. Reciprocation about ω can also be defined entirely synthetically. Let P be a point lying inside ω. Let m be the line through P perpendicular to OP. Let A and B be the intersection points of m and C, and let the tangent lines to ω at A and B intersect at Q. Then the line l through Q perpendicular to OQ is the polar of the point P (and P is the pole of l). In fact, Q and m form a reciprocal pair as well, which gives us a synthetic construction in the case when the point is outside the circle as well. [2]

Since right triangles OPB and OBQ are similar, we see that the distance from l to O is indeed equal to $r^2/|OP|$. (See Figure 4.)

It is clear that reciprocation about ω is a bijection between the points and lines of projective space. We now show that reciprocation is incidence-preserving.

Proposition 1. *Suppose P lies on line q. Let l be the polar of P and let Q be the pole of q with respect to a circle ω having center O and radius r. Then Q lies on line l.*

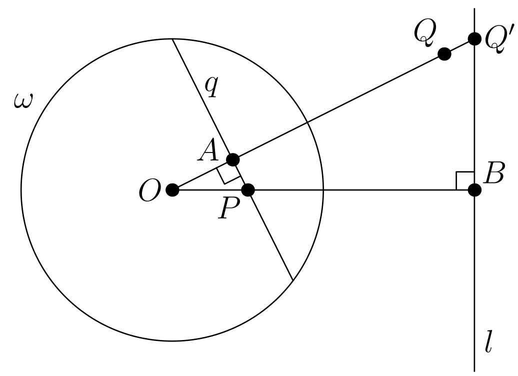

Figure 5: Proving that reciprocation preserves incidence.

Proof. We give the proof in the case that $P \neq O$ is inside circle ω and q does not pass through the center O. The case when P is outside the circle is analogous, and the degenerate cases when P or q intersect O or l^∞ can be easily checked.

Let B be the foot of the perpendicular from O to l. Then O, P, and B are collinear, and we have that $|OP| \cdot |OB| = r^2$.

Let A be the foot of the perpendicular from O to q. Then O, A, and Q are collinear, and we have that $|OA| \cdot |OQ| = r^2$.

Now, let Q' be the point of intersection of line OA with line l. We wish to show that $Q' = Q$.

We have that triangles $\triangle OAP$ and $\triangle OBQ'$ are right triangles sharing a (non-right) angle at O, and are therefore similar. It follows that $|OA| \cdot |OQ'| = |OP| \cdot |OB| = r^2$. Thus $|OA| \cdot |OQ'| = |OA| \cdot |OQ|$, and so $|OQ'| = |OQ|$. Since O, Q, and Q' are collinear and Q and Q' lie on the same side of O, we deduce that $Q = Q'$. □

There is also a beautiful coordinate interpretation of reciprocation in a circle, which naturally generalizes to arbitrary conics. As we have seen, a line in projective space can be described by the homogeneous linear equation $ax + by + cz = 0$. Thus there is a natural isomorphism from $\mathbb{RP}^2$ to $(\mathbb{RP}^2)^*$ sending $(a : b : c)$ to the line $ax + by + cz = 0$.

What does this isomorphism look like geometrically? Let $A = (a : b : c)$ be a point on S^+. Then the equation $ax + by + cz = 0$ defines the plane through the origin normal to RA where $R = (0, 0, 0)$ is the center of the unit sphere in $\mathbb{R}^3$. This plane out a great circle on the unit sphere, which projects under gnomonic projection to a line l on the plane $z = 1$.

Let P be the image of A under gnomonic projection onto $z = 1$, and let $O = (0, 0, 1)$ be the origin in this plane. Let Q be the intersection of line PO with l. Then since l lies in the plane perpendicular to RP, we have that angle PRQ is a right angle. By symmetry, we also see that Q must be the foot of the perpendicular from H to line l, and that PQ is perpendicular to l as well. We obtain the diagram in Figure 6.

Since triangles QOR and POR are similar, and $|OR| = 1$, we have that the distance of l from O, namely $|QO|$, is the reciprocal of the distance $|PO|$. So, the map $(a : b : c) \mapsto (ax+by+cz = 0)$ *almost* corresponds to reciprocation in the unit circle, but the image of P is on the wrong side of O. By reflecting the plane $ax + by + cz = 0$ in the x-y plane (i.e. negating the z coordinate),

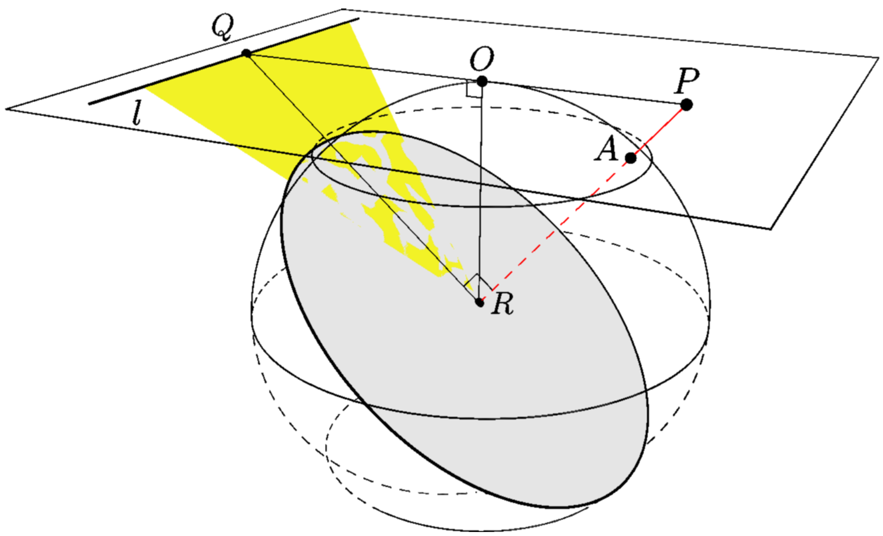

Figure 6: The duality sending $(a : b : c)$ to the line $ax + by + cz = 0$.

we see that the reciprocal of P in the unit circle $x^2 + y^2 = 1$ is the line described by the equation $ax + by - cz = 0$.

The minus sign in the z coordinate is not a coincidence; it arises naturally by considering the projectivization of the equation describing the unit circle on the plane $z = 1$, namely $x^2 + y^2 - z^2 = 0$. With this in mind, we can now define reciprocation with respect to any conic.

Definition 7. Let σ be an arbitrary conic in $\mathbb{RP}^2$, defined by

$$M_{xx}x^2 + M_{yy}y^2 + M_{zz}z^2 + 2M_{xy}xy + 2M_{xz}xz + 2M_{yz}yz = 0. \tag{1}$$

Then *reciprocation* in σ is the isomorphism $\mathbb{RP}^2 \to (\mathbb{RP}^2)^*$ sending the point $(a : b : c)$ to the line $rx + sy + tz = 0$ where r, s, and t are defined by

$$\begin{pmatrix} M_{xx} & M_{xy} & M_{xz} \\ M_{yx} & M_{yy} & M_{yz} \\ M_{zx} & M_{zy} & M_{zz} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} r \\ s \\ t \end{pmatrix}.$$

For a synthetic description of reciprocation in a general conic, we refer the reader to [2].

For our purposes, to understand reciprocation in a conic, we first recall some facts from the theory of quadratic forms. Notice that the conic 1 is described by the quadratic form defined by the real symmetric matrix $M = (M_{ij})$, where $i, j \in \{x, y, z\}$. It is well-known that any real symmetric matrix $<$ can be written in the form $M = A^T JA$ where J is one of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

In other words, upon "completing the square", every conic is equivalent to one of $x^2 + y^2 + z^2$, $x^2 + y^2 - z^2$, $x^2 - y^2 - z^2$, or $-x^2 - y^2 - z^2$ via a linear transformation of the homogeneous coordinates, called a *projective transformation.*

Definition 8. A *projective transformation* is a map $\phi : \mathbb{RP}^2 \to \mathbb{RP}^2$ of the form

$$(x : y : z) \mapsto (ax + by + cz : dx + ey + fz : gx + hy + iz)$$

for some real $a, b, c, d, e, f, g, h, i$.

Notice that $x^2 + y^2 + z^2 = 0$ and $-x^2 - y^2 - z^2 = 0$ describe the same set of points, and similarly $x^2 + y^2 - z^2$ and $-x^2 - y^2 + z^2$ are also equivalent, so every conic is equivalent under a projective transformation to one of $x^2 + y^2 + z^2 = 0$ or $x^2 + y^2 - z^2 = 0$. The former, however, has no solutions in the real projective plane. Thus we have obtained the following fact.

Proposition 2. *Every nondegenerate conic in the real projective plane is equivalent under a linear projective transformation to the circle $x^2 + y^2 - z^2 = 0$.*

In terms of the Euclidean plane, given a nondegenerate conic described by a quadratic equation $f(x, y) = 0$, we can always complete the square and make a linear change of variables to transform it to the unit circle $x^2 + y^2 = 1$. For this reason, in the next section we always use "reciprocation" to mean reciprocation in the unit circle centered at the origin.

3 Duality of plane algebraic curves

An *algebraic curve* in $\mathbb{RP}^2$ can be defined as the locus of points $(x : y : z)$ satisfying a homogeneous polynomial equation $P(x, y, z) = 0$, that is, a polynomial all of whose terms have the same degree d. (Note: it is easy to verify that if (x, y, z) satisfies P, then so does $(x\lambda, y\lambda, z\lambda)$ for any $\lambda \neq 0$, and so it makes sense to say that $(x : y : z)$ satisfies $P(x, y, z) = 0$.)

However, in order to use the geometric notion of reciprocation in the plane, we instead define an algebraic curve as the projective closure of an affine curve.

Definition 9. An *algebraic curve* is a set of points in $\mathbb{RP}^2 = \mathbb{R}^2 \sqcup l^\infty$ defined by a polynomial equation $f(x, y) = 0$ of degree $d > 0$ as follows. Let S be the locus of points $(x, y) \in \mathbb{R}^2$ satisfying $f(x, y) = 0$, and let T be the set of points on l^∞ corresponding to the linear factors of the homogeneous polynomial consisting of only the degree-d terms of f. Then the algebraic curve defined by $f(x, y)$ is $S \sqcup T$.

Remark 3. We can see that this is equivalent to the definition in terms of homogeneous coordinates by considering the inverse image of an algebraic curve under the gnomonic projection. Writing the points on S^+ in homogeneous coordinates, we have that the point (x, y) satisfying $f(x, y) = 0$ in the plane $z = 1$ corresponds to the point $(x : y : 1)$, which then satisfies the *homogenization* of f: the polynomial formed by multiplying each term in f of degree $k < d$ by z^{d-k} to obtain a homogeneous polynomial $\widetilde{f}(x, y, z)$. For instance, the homogenization of the polynomial $f(x, y) = x^3 + xy + y^2 + 1$ is the homogeneous polynomial $\widetilde{f}(x, y, z) = x^3 + xyz + y^2 z + z^3$.

Then, the points on l^∞ on the algebraic curve defined by f are simply the linear factors of $\widetilde{f}(x, y, 0)$. These correspond to the points of the form $(x : y : 0)$ satisfying $\widetilde{f}(x, y, z) = 0$.

We are nearly ready to define the dual curve. We can loosely define it as follows.

Definition 10. The *dual* of an algebraic curve is the set of all poles of the tangent lines to the curve.

To make this precise, we need to define a tangent line. Algebraically, the tangent line to a curve C defined by $f(x, y, z) = 0$ at the nonsingular point $(a : b : c)$ is the line $\frac{\partial f}{\partial x}(a, b, c)x + \frac{\partial f}{\partial y}(a, b, c)y + \frac{\partial f}{\partial z}(a, b, c)z$. Its pole with respect to the unit circle is the point

$$\left(\frac{\partial f}{\partial x}(a, b, c) : \frac{\partial f}{\partial y}(a, b, c) : -\frac{\partial f}{\partial z}(a, b, c) \right).$$

If $(a : b : c)$ is a *singular point*, that is,

$$\frac{\partial f}{\partial x}(a, b, c) = \frac{\partial f}{\partial y}(a, b, c) = \frac{\partial f}{\partial z}(a, b, c) = 0,$$

then the formula above for the tangent line results in the pole $(0 : 0 : 0)$, which is not well-defined. To rectify this, assume without loss of generality that $c \neq 0$ and rescale so that $c = 1$. Then, a tangent line to the curve at $(a : b : 1)$ is defined to be (the homogenization of) any linear factor of the polynomial

$$\sum_{i+j=m} \frac{\partial^m f}{\partial x^i \partial y^j} \frac{(x-a)^i (y-b)^j}{i!j!} \tag{2}$$

where m is the smallest integer such that this sum is nonzero.

Geometrically, we can view the tangent line to a curve at point P as the limiting line of the lines AP as A approaches P along the curve. In the case that P is a singularity, we can consider a local parameterization of the curve by a real parameter t, given by a map $P(t)$ such that $P = P(t_0)$. Then, the left-hand limit of the line $P(t)P(t_0)$ will be well-defined in a small interval of values of t even at singularities; there are only a finite number of singularities, so the curve is smooth in a small neighborhood to the left of each singularity in terms of t.

For instance, consider the algebraic curve defined by $y^2 - x^2 - x^3 = 0$. This is a nodal cubic, the graph of which is shown at left in Figure 7. To compute its tangent lines at $(0, 0)$

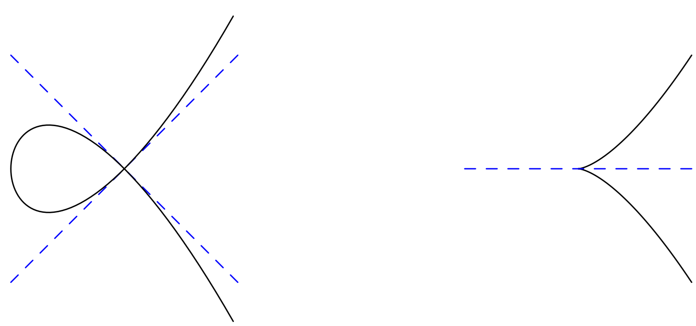

Figure 7: Tangent lines at nodes (left) and cusps (right).

algebraically, we note that the polynomial (2) is $1 \cdot y^2/2! + 1 \cdot x^2/2! = \frac{1}{2}(y-x)(y+x)$, so the tangent lines are $y = x$ and $y = -x$ as shown. Geometrically, the curve is parameterized by $x = t^2 - 1$, $y = t(t^2 - 1)$, and so we can compute the tangent lines as the limits of the lines connecting $(0, 0)$ to $(x(t), y(t))$ as t approaches ± 1.

3.1 A geometric proof that the dual deserves its name

It can be shown that the dual of an algebraic curve is itself algebraic, and we can therefore take the dual of the dual curve. For a proof of this fact, we refer the reader to [1] or [7].

Here, we give a geometric proof of the remarkable fact that taking the dual twice results in the original curve. As usual, we take our conic of reciprocation to be the unit circle centered at the origin.

Theorem 1. *For any algebraic curve C, we have $(C^*)^* = C$.*

Proof. We first show that $C \subset (C^*)^*$. Let P be a point on C. We wish to show that the polar of P, which we call l, is tangent to C^*. Let q be a tangent line to C at P. We claim that its pole Q, which by definition is a point on C^*, is a point of tangency of l to C^*.

We first consider the case that P is not O or on l^∞.

Since reciprocation preserves incidence (Proposition 1), and P lies on q, we know that l passes through Q. To show it is tangent, we wish to show that there is a path in C^* such that as $R \to Q$ along this path, the line QR approaches line l. That is, the angle θ between lines QR and l approaches 0 as R approaches Q along the path.

To show this, we use the fact that q is tangent to C at P. This means that there is a path on C such that as point S approaches P along this path, the line PS approaches q. Let S be a point on this path that is close enough to P such that there are no singularities along the path connecting S to P except perhaps at P, and for simplicity, choose S such that the tangent line r' to C at any point S' on the path between S and P intersects q at a point not at O or infinity. (Note that since the curve C is algebraic, the tangent line at S also approaches the tangent line at P as $S \to P$ since they approach SP uniformly in the distance from S to P.)

Let r be the tangent line to C at S, and let R be the pole of r. Then R lies on C^*, and as $S \to P$, we have $R \to Q$. We wish to show that QR approaches l.

Let M be the intersection point of q and r. Then the polar of M is the line containing Q and R, namely the line QR. It follows that line OM is perpendicular to QR. Let A be their point of intersection. We also have that line OP is perpendicular to l; let B be their point of intersection. Then we have that angles QAO and QBO are right, and so points A and B both lie on the circle with diameter QO. In particular, points Q, A, O, and B all lie on a common circle.

In a cyclic quadrilateral, opposite angles are supplementary and two angles subtended by the same arc are equal. Thus, no matter the order of Q, A, O, and B around the circle, we see that the angle θ formed by $QR = QB$ and $l = QA$ is equal to the angle QOA, which is the same as angle MOP. But as $S \to P$, we have that $\angle MOP$ approaches 0, and so θ approaches 0 as $R \to Q$, as desired.

We now need to consider the degenerate cases when P is at O or on l^∞. We wish to show that its polar l^∞ is tangent to C^*. Let q and Q be as before; now Q lies on l^∞. We can define S, r, R, and M as before; now, as $S \to P$ we have that $M \to P$ and hence QR, the polar of M,

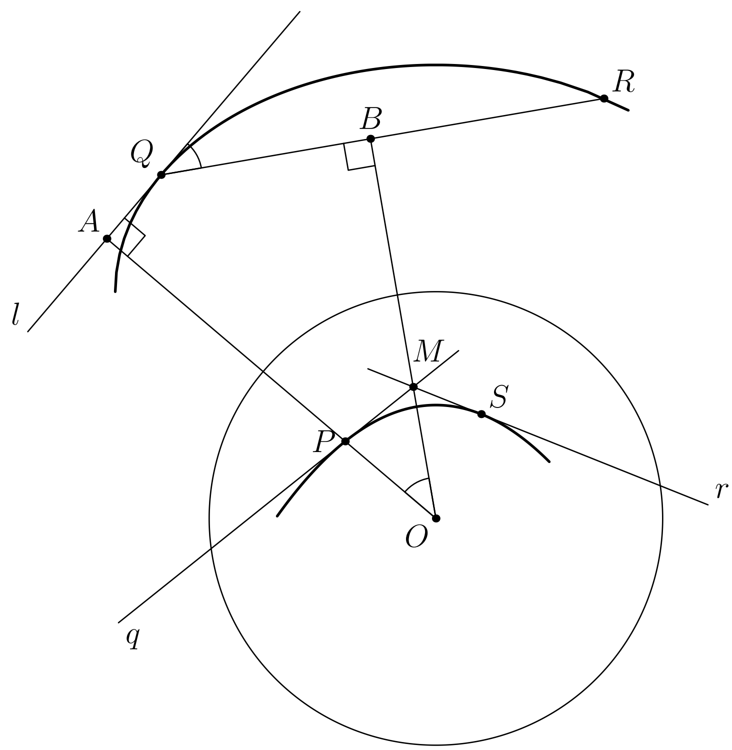

Figure 8: Proving that $(C^*)^* = C$.

is a line perpendicular to q whose distance to the origin approaches ∞; thus, QR approaches l^∞.

In the case that P is on l^∞, the "tangent line" q to C at P is either l^∞ or is an asymptote of the curve. If q is l^∞, then as $S \to P$ we have that the tangent line at S corresponds to a point M at infinity that approaches P, whose polar is a line passing through O that approaches the direction of the polar l of P. Thus l is tangent to C^* at $Q = O$. If instead q is an asymptote of the curve, then we consider a point S on the curve approaching infinity along the asymptotic path, and its tangent lines will intersect q at a point M approaching infinity (P) along the line. Then $\angle MOP$ is well-defined in this case, and as in the first case it is equal to the angle between QR and the polar l of P, where R is the pole of the tangent line to C at S.

We have now shown in all cases that $C \subset (C^*)^*$. For the other inclusion, $(C^*)^* \subset C$, let P be a point in $(C^*)^*$. Let l be its polar. Then l is tangent to C^* at a point Q. Let q be the polar of Q. Then q is tangent to C at some point; we wish to show that one of these points P' coincides with P. To do so, we wish to show that the polar of one of the points of tangency P' of q to C is l. In other words, we need to show that the polar of P' is tangent to C^* at Q. We can then use the same proof as above, choosing P' to be the point of tangency corresponding to the tangent line l upon taking the dual of C. Thus $P = P'$, and so P lies on C. □

3.2 Properties of the Dual Curve

Consider the dual of a curve C at a node P (Figure 7, left). Since there are multiple tangents to the point, their poles give several points on the dual curve. By Theorem 1, each of these points lie on the polar of P, and in fact the polar of P is tangent to C^* at each of the points. We call a line that is tangent to a curve at two points a *bitangent*.

Now, suppose P is a cusp of C. By a similar analysis, there is a corresponding *inflection point* on the dual curve.

Proposition 3. *Let C be an algebraic curve. There is one bitangent on C^* for each node on C, and one inflection point on C^* for each cusp on C.*

These singularities are essential for understanding the relationship between the degree of a curve and its dual. We say that a singularity is an *ordinary double point* if it is a node with exactly two crossings, that is, it looks locally like the curve xy at $(0,0)$. A singularity is a *simple cusp* if it looks locally like the curve $y^2 = x^3$ at $(0,0)$. (See Figure 7.) We also say that an inflection point is simple if it corresponds to a simple cusp on the dual curve.

Proposition 4 (Plücker's formula)**.** *Let C be an algebraic curve of degree n having only ordinary double points and simple cusps as singularities. Suppose its dual C^* also has only ordinary double points and simple cusps as singularities, or equivalently, that every tangent line to C is tangent at at most two points and its inflection points are simple. Then the degree n' of C^* is given by*

$$n' = n(n-1) - 2d - 3s$$

where d is the number of ordinary double points and s is the number of singularities on C.

This formula was later generalized to arbitrary curves having singularities of any type by Weierstrass and M. Noether. We refer the reader to [1], p. 586 for this generalization.

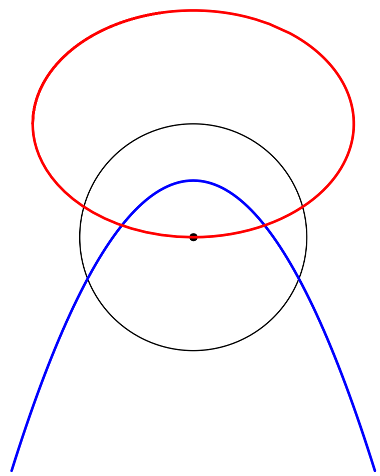

Figure 9: The dual conics $y = \frac{1}{2} - x^2$ (blue) and $(y-1)^2 + \frac{1}{2}x^2 = 1$ (red).

4 Examples

To illustrate some of the properties mentioned in the previous section, we consider a few examples. First, notice that by Plücker's formulae, the dual of a conic (degree 2 curve) is always a conic, and that conics are nonsingular. An example of dual conics is shown in Figure 9.

By the Plücker formulae, the dual of a nonsingular cubic has degree 6, but if the cubic has a node then its dual has degree 4, and if it has a cusp then its dual has degree 3. We demonstrate examples of these in Figures 10 and 11.

Finally, consider a cuspidal cubic with its cusp at the origin, say $y^2 = ax^3$ for some real a. This has an inflection point at infinity with l being its tangent line, and so its dual will also be a cubic (by Plücker's formula) and will have a cusp at the origin and an inflection point at infinity.

This suggests that there may be such a curve whose dual yields the original curve; we call such a curve *self-dual*. In general, any curve that is projectively equivalent to its dual is said to be self-dual, and the problem of classifying the self-dual real plane algebraic curves is currently open. [3]

Note that since all conics are projectively equivalent, if a curve is projectively equivalent to its dual then we can choose a conic σ such that the curve reciprocates to itself with respect to σ. We can then apply the projective transformation sending σ to the unit circle ω to obtain an equivalent curve that is self-dual with respect to ω. Therefore, in order to understand self-dual curves, it suffices to understand those curves which reciprocate to themselves in the unit circle.

In order to find a self-dual curve, we compute the dual of the curve C given by $y^2 = ax^3$ where $a > 0$ is a real constant. The curve has a parameterization given by

$$x(t) = t^2$$

and

$$y(t) = \sqrt{a}t^3.$$

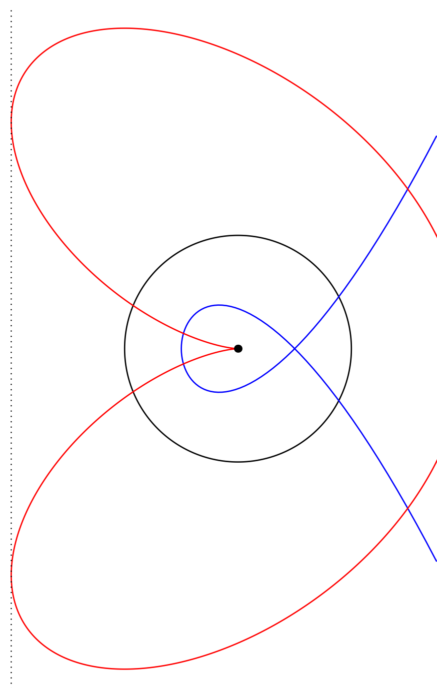

Figure 10: The cubic curve $y^2 = (x - 1/2)^2(x + 1/2)$, shown in blue, and its dual shown in red. The polar of the node on the blue curve is the bitangent to the red curve as shown by the dotted green line.

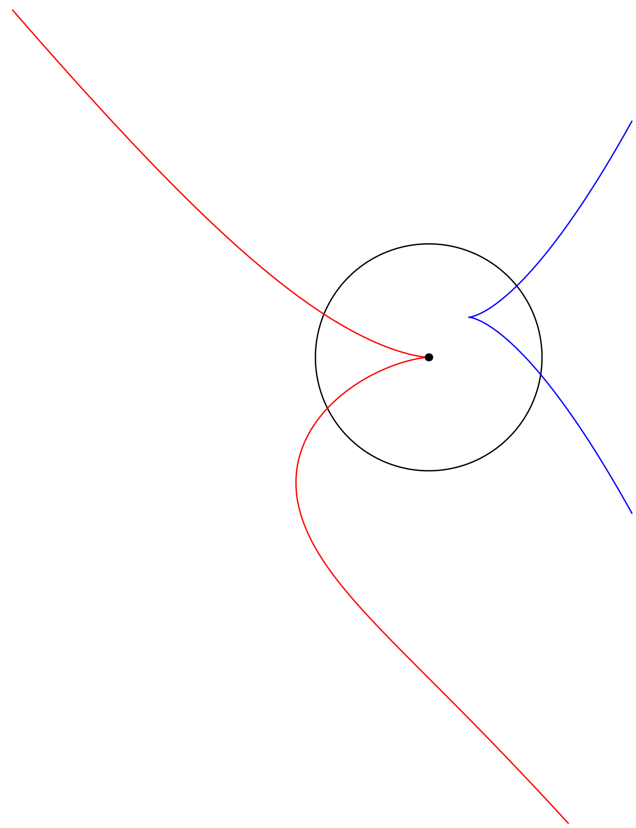

Figure 11: The cubic curve $(y - \sqrt{2}/4)^2 = (x - \sqrt{2}/4)^3$, shown in blue, and its dual shown in red. The dual has an inflection point at infinity, and the tangent line to this inflection point is the polar of the cusp of the blue cubic.

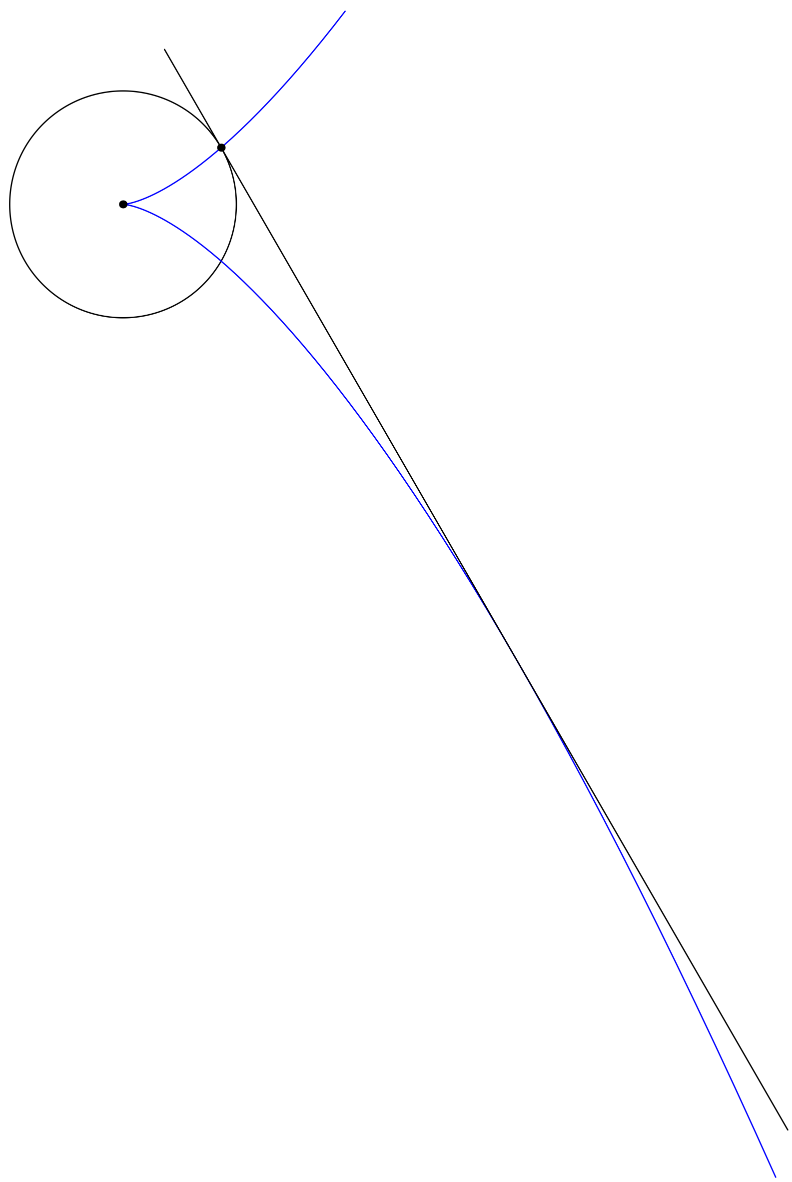

Figure 12: The self-dual cubic curve $y^2 = \sqrt{\frac{4}{27}}x^3$. The self-duality is exhibited here by the fact that the common tangent line shown to the cubic and the circle is tangent to the circle at another point on the cubic.

Then the tangent line to $(x(t_0), y(t_0))$ is given by the equation

$$-y'(t_0)x + x'(t_0)y = -y'(t_0)x(t_0) + x'(t_0)y(t_0),$$

and so the points on C^* correspond to the poles of these points:

$$\left(\frac{-y'(t)}{y'(t)x(t) - x'(t)y(t)}, \frac{x'(t)}{y'(t)x(t) - x'(t)y(t)}\right).$$

Thus in this case the dual curve is parameterized by

$$x^*(t) = 3/t^2$$

and

$$y^*(t) = 2/(\sqrt{a}t^3),$$

which satisfy $y^2 = \frac{4}{27a}x^3$.

Thus, the dual curve is the same as the original curve if $\frac{4}{27a} = a$, which occurs when $a = \sqrt{\frac{4}{27}}$.

5 Acknowledgments

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