

Math 672: Projective Geometry I

Homework 5

Hand in between 2 and 5 inclusive of the problems below; mark at the top of your page which problem numbers you are handing in. Write full proofs for each of your solutions. Your score will be as a percentage of those you hand in. To submit your solutions, either type on Overleaf/LaTeX or handwrite and take pictures for file uploads, and upload them to the Canvas course under the appropriate Assignment.

Problems

1. (Shafarevich Book 1 ch 1 sec 5 problem 1) Prove that the Segre variety $\mathbb{P}^n \times \mathbb{P}^m \subseteq \mathbb{P}^N$ (where $N = (n+1)(m+1) - 1$) is not contained in any projective linear subspace strictly smaller than the whole of $\mathbb{P}^N$. (That is, show it does not satisfy any linear homogeneous equation in the variables of $\mathbb{P}^N$).
2. (Shafarevich Book 1 ch 1 sec 5 problem 2) Consider the two maps of varieties $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ given by $p_1(x, y) = x$ and $p_2(x, y) = y$, in other words, the two projection maps from the product. Prove that $p_1(X) = p_2(X) = \mathbb{P}^1$ for any closed irreducible subset X of $\mathbb{P}^1 \times \mathbb{P}^1$, unless X is one of the following types: (a) a point $(x_0, y_0) \in \mathbb{P}^1 \times \mathbb{P}^1$, or (b) a line $x_0 \times \mathbb{P}^1$ or $\mathbb{P}^1 \times y_0$ for some x_0 or y_0 .
3. We saw in class that the Segre variety embeds $\mathbb{P}^m \times \mathbb{P}^n$ into $\mathbb{P}^{(n+1)(m+1)-1}$. If we iterate this process on a product of three projective spaces instead of two, what does $\mathbb{P}^m \times \mathbb{P}^n \times \mathbb{P}^\ell$ embed into?
4. The variety of all flags $0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = k^n$ in n dimensional space over k , which consist of chains of subspaces V_i where $\dim V_i = i$ for all i , can be embedded as a projective subvariety of the product of Grassmannians $\text{Gr}(1, n) \times \text{Gr}(2, n) \times \cdots \times \text{Gr}(n, n)$. By using the Plücker embedding for each Grassmannian and combining it with an iterated Segre embedding, what dimension projective space is the flag variety embedded into?
5. Consider the chain of sets

$$\{1\} \subseteq \{1, 2\} \subseteq \{1, 2, 3\} \subseteq \cdots$$

where we think of each subset symbol as the natural inclusion map. What is the direct limit of this system?

6. Consider the chain of symmetric groups

$$S_1 \hookrightarrow S_2 \hookrightarrow S_3 \hookrightarrow \cdots$$

where each symmetric group S_n , thought of as the set of all permutations of $1, \dots, n$, naturally embeds into S_{n+1} by simply considering the permutations that fix $n+1$. What is the direct limit of this system?

7. Consider the sequence of projection maps formed by forgetting the last coordinate to map $\mathbb{C}^n \rightarrow \mathbb{C}^{n-1}$ for all n . This gives a chain:

$$\cdots \rightarrow \mathbb{C}^4 \rightarrow \mathbb{C}^3 \rightarrow \mathbb{C}^2 \rightarrow \mathbb{C}^1.$$

What does the inverse limit look like in the category of vector spaces? Describe its elements.

8. Consider the sequence of inclusions of vector spaces

$$\mathbb{C}^1 \hookrightarrow \mathbb{C}^2 \hookrightarrow \mathbb{C}^3 \hookrightarrow \mathbb{C}^4 \hookrightarrow \cdots$$

by embedding $\mathbb{C}^n$ as spanned by the first n coordinates of $\mathbb{C}^{n+1}$. What does the direct limit look like in the category of vector spaces? Describe its elements.

9. (Shafarevich book 2 chapter 5 section 2 problem 1) Let X be a discrete topological space and $F(U)$ the set of all maps $f : U \rightarrow M$ such that $f(U)$ is finite; for $U \subseteq V$, ρ_U^V is restriction. Is F a presheaf? A sheaf?