Math 672: Projective Geometry I
Homework 3

Hand in between 2 and 5 inclusive of the problems below; mark at the top of your page which problem numbers you are handing in. Write full proofs for each of your solutions. Your score will be as a percentage of those you hand in. To submit your solutions, either type on Overleaf/LaTeX or handwrite and take pictures for file uploads, and upload them to the Canvas course under the appropriate Assignment.

Problems

1. Consider ideal $J = (y - x^2, y - 2x + 1) \subseteq \mathbb{C}[x, y]$.
   (a) To gain intuition, draw $y - x^2 = 0$ and $y - 2x + 1 = 0$ as curves over $\mathbb{R}$. What do you notice about their intersection point(s)?
   (b) Compute the radical ideal $\sqrt{J}$.
   (c) What is $V(J)$? What is $V(\sqrt{J})$?
   (d) What is $\operatorname{Spec}(\mathbb{C}[x, y]/J)$? What is $\operatorname{Spec}(\mathbb{C}[x, y]/\sqrt{J})$? Explain how they differ. How many prime ideals are in each?
2. **Real algebraic geometry:** (warning: the real numbers are not algebraically closed!) What do the prime ideals look like in $\operatorname{Spec}(\mathbb{R}[x])$? Explain how this means that $\operatorname{Spec}(\mathbb{R}[x])$ captures a generic point, the closed points of the real line, and complex conjugate pairs in an imaginary complex plane outside of it.
3. Prove that the cartesian product of two sets does give their category theoretic product (as a universal object) in the category of sets.
4. **Union of varieties:** Let X be the union of the algebraic curves defined by $x^2 + y^2 = 1$ and $y = x$. Let I_X be its defining ideal, and consider $\operatorname{Spec}(\mathbb{C}[x, y]/I_X)$. Identify the prime ideals that function as the generic point corresponding to each irreducible component of X.
5. **Intersection of varieties:** Let X be the intersection of the algebraic curves defined by $x^2 + y^2 = 1$ and $y = x$. Let I_X be its defining ideal and consider $\operatorname{Spec}(\mathbb{C}[x, y]/I_X)$. How many points does it have?
6. Shafarevich ch 5 section 1 problem 1 - Let N be the nilradical of a ring A and $\varphi : A \to A/N$ the quotient map. Prove that the associated map ${}^a\varphi : \operatorname{Spec}(A/N) \to \operatorname{Spec} A$ is a homeomorphism.
7. Shafarevich ch 5 section 1 problem 5: Prove that $\overline{{}^a\varphi(V(E))} = V(\varphi^{-1}(E))$ where the bar means closure.
8. Show that the identity morphism in any object in a category is unique.
9. Exercise 1.2.C in Vakil's notes https://math.stanford.edu/ vakil/216blog/FOAGoct2125public.pdf

   Note: you will have to read some of the notes before this problem to understand it; this is a "going above and beyond" sort of exercise, but can help understand localization.
10. Exercise 1.2.G in Vakil's notes