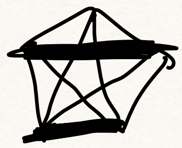
Hall's Lemma, Matchings

<u>Def</u>: A <u>matching</u> in an undirected graph $G=(V,E)$ is a subset S of E s.t. no two of the edges in S have a common vertex:

<u>Ex</u>:

matching



matching

<u>maximal</u> matching

<u>Maximal</u> <u>matching</u>: can't add any edges to make a larger matching.

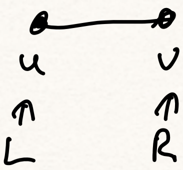
Two maximal matchings:

maximal

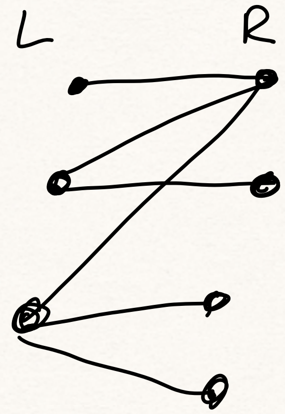
vs

maximum

Def: A graph is <u>bipartite</u> if V can be partitioned into two sets $L \cup R$ s.t. all edges are



Ex:



Fact: All trees are bipartite! (why)?

Fact: A graph is bipartite iff it has no odd cycles (Hwk)

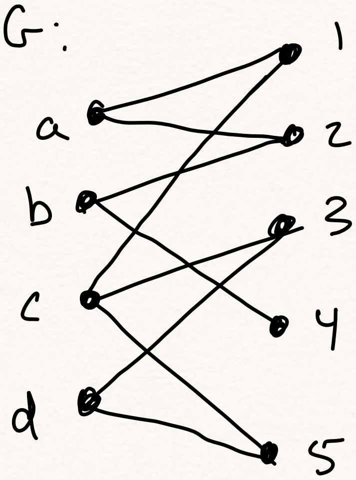
Def: Say G bipartite, $|L| \leq |R|$.

A matching is <u>saturated</u> if it uses all of L.

<u>Neighborhood</u> of v in G: $N_G(v) = \{u : \exists \text{ edge } u - v\}$

- $N_G(S) = \bigcup_{v \in S} N_G(v)$.

Ex: G:



$N_G(\{a,b\}) = \{1,2,4\}$.

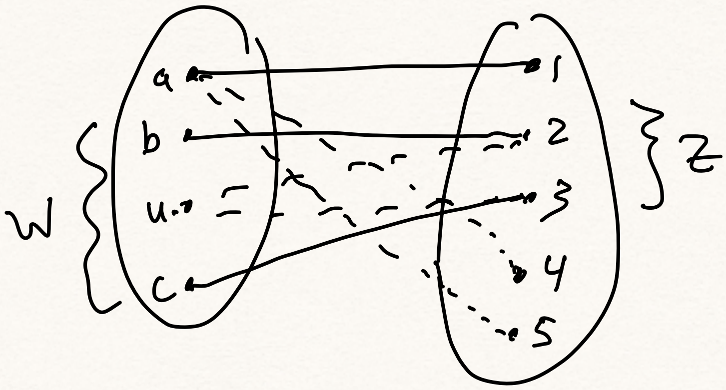
Hall's Marriage Lemma: If G bipartite w/ $|L| \leq |R|$, then G has a saturated matching iff $\forall S \subseteq L$, $|N_G(S)| \geq |S|$.

Pf: ($\Rightarrow$) Saturated matching $\varphi: L \to R$ means $\forall S \subseteq L$,

$$|N_G(S)| \geq |\varphi(S)| = |S| . \checkmark$$

($\Leftarrow$) Suppose ineq. holds and, for contradiction, assume no sat. matching. Let M be largest-possible matching, $u \in L$ unmatched.

Consider all alternating paths (unmatched, matched, unmatched, matched...) starting @ u.



$W = \{$vertices in L among alt. paths from $u\}$

$Z = \{$vertices in R among alt. paths from $u\}$

Claim 1: $N_G(W) = Z$.

Pf of claim: If $z \in Z$, adj to some elt of W

$\Rightarrow$ $Z \subseteq N_G(W)$

Now, let $z \in N_G(W)$; adj to some $v \in W$.

Consider alt. path $u \to v$; ends w/ match.

If v matched to z then $v - z$ is end of $u \to v$,

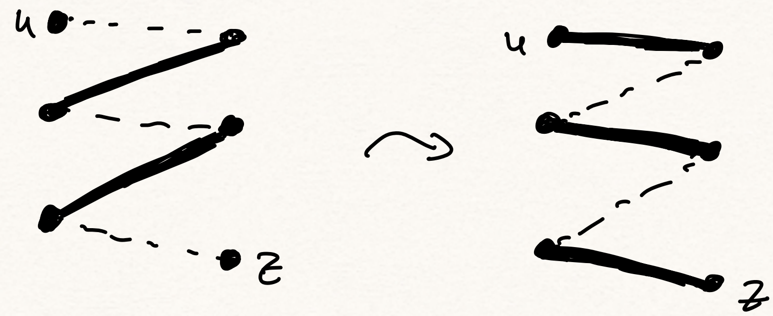
else it extends the path

$\Rightarrow z \in Z$.

So $N_G(W) \subseteq Z$.

Claim 2: $W \geq |Z| + 1$.

Pf of claim: Every vertex in Z matched to something in W, since otherwise a path can be flipped to increase $|M|$:



Since u is unmatched, $|W| \geq |Z| + 1$.

The two Claims give a contradiction.

QED