Math 501: Combinatorics
Homework 5

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

Note: To get the most out of this homework learning-wise, I recommend that no more than 6 of your points come from the problems involving the formalities of generating functions (problems 1,2,3). And remember to prove all your answers below!

Problems

1. (1+) (2 points) Show that multiplication of formal power series is associative. That is, if $A(x)$, $B(x)$ and $C(x)$ are three generating functions, show that $(A(x)B(x))C(x) = A(x)(B(x)C(x))$. Do so by the formal rules for multiplication that we gave in class, and use the fact that two generating functions are equal if and only if their sequence of coefficients are equal.

2. (1+) (2 points) Show that the product rule for differentiation holds for formal power series. That is, show that if $A(x)$ and $B(x)$ are two generating functions, then

$$\frac{d}{dx}(A(x)B(x)) = \left(\frac{d}{dx}A(x)\right)B(x) + A(x)\left(\frac{d}{dx}B(x)\right).$$

   Do this using the formal definitions of derivative and multiplication that we gave in class.

3. (2+) (4 points) Show that the chain rule holds for formal power series. That is, show that if $G(x)$ has no constant term and $F(x)$ is any generating function, then $\frac{d}{dx}F \circ G(x) = \frac{d}{dx}F(G(x)) \cdot \frac{d}{dx}G(x)$.

4. (1+) (2 points) Prove the generating function identity

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \left(\!\!\binom{n}{k}\!\!\right) x^k.$$

   You may either use induction on n, or a direct combinatorial argument about what the coefficients must be when you expand the product on the left.

5. (1+) (2 points) Find a closed form for the generating function of the sequence $a_n = n^2$, that is, find a closed form for

$$\sum_{n=0}^{\infty} n^2 x^n.$$

6. (2) (3 points) Find a closed form for the generating function of the sequence b_n defined by $b_0 = 1$ and for all $n \geq 0$, $b_{n+1} = \sum_{k=0}^{n} k b_{n-k}$. Use it to find an explicit formula for b_n in terms of n.

7. (2) (3 points) Let R_n denote the number of permutations $\pi \in S_n$ for which $\pi_1 = 1$, $\pi_n = n$, and $|\pi_i - \pi_{i+1}| \leq 2$ for all $i \leq n-1$. Show that $R_0 = 0$, $R_1 = 1$, $R_2 = 1$, and for all $n \geq 3$,

$$R_n = R_{n-1} + R_{n-3}.$$

   Use this recursion to find a closed form for the generating function of the sequence R_n.

8. (2) (3 points) Let $p(n,k)$ be the number of partitions of n into exactly k nonzero parts. Show that

$$\sum_{n,k} p(n,k) y^k x^n = \prod_{k=0}^{\infty} \frac{1}{1-yx^k}.$$

9. (2-) (3 points) Use generating functions to prove that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Do NOT give a combinatorial proof. Instead, give a proof by comparing coefficients of two equal generating functions or polynomials.